Upper and lower bounds of stochastic resonance and noise-induced synchronization in a bistable oscillator

Agnessa Kovaleva
Russian Academy of Sciences, Space Research Institute, Moscow 117997, Russia
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This paper discusses concepts of stochastic resonance and noise-induced synchronization in a bistable oscillator subject to both periodic signal and noise. We demonstrate that stochastic resonance is not directly correlated with the matching of the signal frequency and the switching rate. The phenomena of stochastic resonance and noise-induced synchronization are the limiting cases of noise-induced transitions, and the spectral response heavily depends on the input signal-to-noise ratio. The lower and upper bounds of noise intensity allowing synchronization are found as functions of the system's parameters.

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I. INTRODUCTION

The concept of stochastic resonance (SR) has been introduced to describe a paradoxical phenomenon in multistable systems excited by a weak periodic signal and noise: an increase in the input noise can result, under certain conditions, in an improvement of the output signal-to-noise ratio (SNR) [1,2]. The simplest mathematical model considers the first-order diffusion in a periodically changing bistable potential landscape. The system's dynamics has been taken to be discrete: a particle exhibits instant random jumps between the bottoms of the potential wells [1]. A weak signal causes the positions of the potential's minima and maximum to oscillate, and interwell jumps seem to occur with some degree of coherence with the signal. The noise intensity can be chosen in such a way that the system's sensitivity and ability to amplify the weak input would be maximal.

Recent theoretical advances in this field [3] are based on large deviation theory [4]. It has been proved [4] that a particle in a bistable potential rests near the bottom of the deeper well with probability close to 1, but with eventual noise-induced excursions to the other well. This results in the above-mentioned discrete model. The rudiments of similar reasonings can be found in [1,2].

The effect of a periodic signal is similar to tilting of the potential. It seems likely for a particle to jump from a higher well to a deeper well at the extrema of the tilting cycles. The existing experimental and simulation results [5] are in a fairly good agreement with this hypothesis. This leads to a commonly used interpretation of SR as a noise-induced synchronization phenomenon. However, synchronization of hopping and signal is not implicit in SR theory. By this theory, the output spectrum is a superposition of a flat wide-band spectrum of the Lorentzian type and a discrete spectrum with a peak at the signal frequency; no coherence between the interwell switching rate and the signal is implicit in this representation.

In this paper we demonstrate that the phenomena of stochastic resonance and noise-induced synchronization are different but not contradictory. SR theory considers weakly modulated systems, with a small input SNR. The signal is insufficient to lock the hopping process, so that random hopping is dominant, and a wide-band component of the spectrum is substantial. As the relative level of the signal against noise is increasing, the wide-band portion of the spectrum decreases, and locking of the hopping process to a weak signal becomes visible. The boundary between the domains of stochastic resonance and synchronization is a function of the input SNR.

In the majority of the previous studies, SR has been studied on the base of the first-order (overdamped) model, in which the inertia force is ignored. The hopping dynamics has been reduced to a simple two-state Poissonian model with the Kramers switching rate. We consider a more realistic model of a bistable oscillator

$$\ddot{x} + \varepsilon\beta\dot{x} + V'(\tau,x) = \varepsilon\sigma W(t), \tag{1}$$

where

$$V(\tau,x) = U(x) - \varepsilon\gamma x \sin\tau, \quad \tau = \omega_s t, \tag{2}$$

$0<\varepsilon\ll 1$ is a small parameter of the system, $U(x)$ is a symmetric double-well potential function, $W(t)$ is a zero-mean Gaussian white noise of unit intensity; the overdot denotes differentiation with respect to time t, the prime denotes partial differentiation with respect to x.

Different physical mechanisms of the SNR increase in the overdamped and oscillatory model have been analyzed in [6]. The overdamped model has been studied under the commonly used assumptions of the hopping dynamics. In the analysis of system (1), the signal frequency ω_s has been presumed close to the system's eigenfrequency and much higher than the rate of escape from the potential well. Under this assumption, the SNR improvement is not associated with interwell hopping. Despite the presence of weak noise, the signal enhancement due to an agreement between the signal's and system's frequencies can be interpreted as the classic resonance effect in system (1) [7].

In this paper we investigate the signal enhancement associated with noise-induced interwell jumps in system (1). This allows us to compare similar effects in the first-order and second-order models.

The asymptotic analysis of system (1) has been performed by the stochastic averaging method [8]. Omitting the proof, in Sec. II we recall the expressions of the mean escape time and the mean escape rate for the second-order oscillator (1).

In Sec. III we compare the SR curves and the optimum noise values for the first-order and second-order models. We show that the two-state model with the Kramers switching rate is convenient for the analysis of noise-induced transitions in system (1). In Sec. IV we analyze the effect of the input SNR. We demonstrate that SR can occur in a weakly-modulated system with a small input SNR. If the input SNR is large enough, a weak signal entails synchronization of noise-induced transitions. The upper and lower bounds of synchronization are derived in Sec. IV.

II. ESCAPE RATE AND THE STATE PROBABILITY FOR THE SECOND-ORDER MODEL

The main ingredients of SR theory are the escape time law and the probability of the system's state in the domain of attraction of the unperturbed fixed point. The asymptotic estimates of the statistical quantities have been obtained by the stochastic averaging method [8]. In this item, we recall the main results of [8].

Consider the nonmodulated system,

$$\ddot{x} + \varepsilon\beta\dot{x} + U'(x) = \varepsilon\sigma W(t), \tag{3}$$

where $U(x)$ is a continuous and twice continuously differentiable even function having the saddle (maximum) point $x_0=0$, $U(0)=0$, and two minima $x_+=a>0$ and $x_-=-a<0$ such that $U(a)=U(-a)<0$. The point $x=0$ corresponds to the saddle point, the points $x_+=a>0$ and $x_-=-a<0$ correspond to the stable centers of the conservative counterpart of system (3) as $\varepsilon=0$. Here and below the indices + and − are related to the right and left half-planes, associated with the centers $x_+=a>0$ and $x_-=-a<0$, respectively. The index ± correspond to the entities defined in both half-planes.

In the phase portrait of the conservative oscillator the domains of attractions $Q_\pm$ of the centers $x_\pm$ are enclosed by the loops of the homoclinic separatrix with the saddle point $x_0=0$ [7]. Let system (3) be within $Q_\pm$ at the initial moment $t_0=0$. If perturbation is weak, escape time $T^\pm$ is interpreted as the time needed to reach the separatrix by the process starting at a point within $Q_\pm$. The requisite statistical parameters are the mean escape time $\Theta^\pm=\mathbf{E}T^\pm$ and the mean escape rate $\lambda^\pm=1/\Theta^\pm$. Here and below the symbol $\mathbf{E}$ denotes expectation of the random variable. In the symmetric system (3) we have $T^\pm=T^\varepsilon$, $\Theta^\pm=\Theta^\varepsilon$.

It has been shown [4] that the asymptotic estimate of escape time T^ε as $\varepsilon\to 0$ is independent of the initial state. It can be calculated as the time needed to reach the potential barrier $U(0)=0$ by a particle starting at the bottom point $\pm a$. Asymptotic approximations of the mean escape time and escape rate from the points $\pm a$ through the threshold $U(0)$ are [8]

$$\Theta^\varepsilon \underset{\varepsilon\to 0}{\to} \theta^0 = K^{-1}\exp(2\beta\Delta U/\varepsilon\sigma^2),$$

$$\lambda_0 = 1/\theta^0 = K\exp(-2\beta\Delta U/\varepsilon\sigma^2), \tag{4}$$

where $K=2\beta^2\omega_0 I_s/\sigma^2$, I_s is the action integral calculated along the loop of the homoclinic separatrix [7], $\Delta U=U(0)-U(\pm a)$, $\omega_0=[U''(\pm a)]^{1/2}$.

Finally, we compare λ_0 with the Kramers rate α_0 calculated for the first-order model. The formal derivation of the Kramers rate has been given for the dimensionless system, in which $\ddot{x}=0, \varepsilon\beta=1$, $\varepsilon\sigma=\sqrt{D}$ [9]. The obvious scaling yields

$$\alpha_0 = \kappa\exp(-2\beta\Delta U/\varepsilon\sigma^2), \tag{5}$$

where $\kappa=\omega_0\omega_1/(\sqrt{2}\pi\varepsilon\beta)$, $\omega_1=|U''(0)|^{1/2}$. Since the factor κ is independent of σ, the rate α_0 is less sensitive to the noise intensity than λ_0. However, we have $\ln\lambda_0\to\ln\alpha_0$ as $\varepsilon\to 0$ on a commonly used logarithmic scale.

Now we consider the modulated system (1), (2). We assume that the signal frequency ω_s satisfies the condition

$$\ln(1/\omega_s)\to k/\varepsilon, \quad k>0, \quad \text{as } \varepsilon\to 0. \tag{6}$$

Assumption (6) implies that the signal period is longer than any time scale of the intrawell dynamics. This allows considering the slow dimensionless time $\tau=\omega_s t$ as a fixed parameter [1–5]. Let $x_0(\tau)$ and $x_\pm(\tau)$ be the maximum and the minima of the potential $V(\tau,x)$ found as the functions of the parameter τ. The modulated escape rates out of the states $x_\pm(\tau)$ are [8]

$$\lambda^\pm(\tau) = \lambda_0^\pm(\tau)\exp(\mp\nu\sin\tau), \quad \nu = 2\beta a\gamma/\sigma^2, \tag{7}$$

where

$$\lambda_0^\pm(\tau) = K^\pm(\tau)\exp\left[\frac{-2\beta\Delta V^\pm(\tau)}{\varepsilon\sigma^2}\right], \quad K^\pm(\tau) = \frac{2\beta^2\omega_0^\pm(\tau)I_s^\pm(\tau)}{\sigma^2}. \tag{8}$$

$V^\pm(\tau)=V(\tau,x_0(\tau))-V(\tau,x_\pm(\tau))$, $\omega_0^\pm(\tau)=[V''(\tau,x_\pm(\tau))]^{1/2}$, $I_s^\pm(\tau)$ are the action integrals calculated along the slowly pulsing loops of the separatrix. The parameter ν is a measure of the input SNR.

It has been proved [8] that $\lambda_0^\pm(\tau)=\lambda_0(1\pm\varepsilon\rho\sin\tau+\varepsilon^2\cdots)$, $\rho\sim a\gamma/\Delta U$. Since the normalized depth of modulation $\varepsilon\gamma a\ll\Delta U$, one can write

$$\lambda^\pm(\tau) = \lambda_0\exp(\mp\nu\sin\tau) \quad \text{as } \varepsilon\to 0. \tag{9}$$

The Fourier decomposition of the periodic rate (9) is

$$\lambda^\pm(\tau) = \lambda_0\left[I_0(\nu) + 2\sum_{n=1}^{\infty}(-1)^n I_n(\pm\nu)\sin n\tau\right], \tag{10}$$

where I_n is the modified Bessel function of order n [10]. From formulas (9), (10) it follows that the precise form of the time-dependent parameters in insubstantial for the asymptotic analysis.

We define the state probability $\mathbf{P}^\pm(t|t_0,\chi)$ as the probability of the system being within the domain Q_+ or Q_- at a moment t after random walks between the wells, provided the process starts at a given point $\chi=a$, or $\chi=-a$. Large deviation theory allows reducing the hopping dynamics in system (3) to a simple Poissonian switching model. In this model, a particle may be either at $+a$ or at $-a$, the passages between these positions are instant. This yields the Poissonian state probability $\mathbf{P}^\pm(t|t_0,\chi)=\mathbf{P}[x(t)\pm a|x(t_0)=\chi]$ governed by the rate equation [9]

$$\dot{\mathbf{P}}^{+} = -\dot{\mathbf{P}}^{-} = -[\lambda^{+}(\tau) + \lambda^{-}(\tau)]P^{+} + \lambda^{-}(\tau), \quad \mathbf{P}^{+} + \mathbf{P}^{-} = 1, \tag{11}$$

with the boundary conditions

$$\mathbf{P}^{+}(t_0|t_0,\chi) = \mathbf{P}[x(t_0) = a|\chi] = \delta_{a,\chi},$$

$$\mathbf{P}^{-}(t_0|t_0,\chi) = \mathbf{P}[x(t_0) = -a|\chi] = \delta_{-a,\chi},$$

where $\delta_{a,\chi}=\{1, \text{ if } \chi=a; 0, \text{ if } \chi=-a\}$ is the Kronecker symbol.

III. STOCHASTIC RESONANCE IN A WEAKLY MODULATED SYSTEM

Following [1], we introduce the main definitions and notations of SR theory. The task is to compare the results for the overdamped Kramers model and second-order oscillatory model and to reveal the discrepancies in the commonly used description of SR effect.

The system is said to be weakly modulated if the input SNR $\nu<1$. In this case we find from (9), (10)

$$\lambda^{\pm}(\tau) \approx \lambda_0(1 \mp \nu \sin \tau), \quad \nu < 1. \tag{12}$$

The output SNR is defined through the averaged power spectrum

$$S(\Omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt \int_{-\infty}^{\infty} R_0(t,s)\cos \Omega s ds, \tag{13}$$

where $R_0(t,s)$ is the correlation function of the steady-state portion of the two-state Poissonian process

$$\begin{aligned} R_0(t,s) &= \lim_{t_0\to-\infty} \mathbf{E}[x(t)x(s)|t_0,\chi] \\ &= a^2[\mathbf{P}^{+}(t|s,a)\mathbf{P}^{+}(s) - \mathbf{P}^{+}(t|s,-a)\mathbf{P}^{-}(s) \\ &\quad - \mathbf{P}^{-}(t|s,a)\mathbf{P}^{+}(s) + \mathbf{P}^{-}(t|s,a)\mathbf{P}^{-}(s)]. \end{aligned} \tag{14}$$

The state probability $\mathbf{P}^{\pm}(t)$ is defined as the steady-state solution of the rate equation (11) as $t_0\to-\infty$, that is

$$\mathbf{P}^{\pm}(t) = \frac{\lambda^{\mp}(t)}{2\lambda_0} = \frac{1 \pm \nu \sin \tau}{2}. \tag{15}$$

Now, by formulas (14) and (15),

$$R_0(s) = a^2\exp(-2\lambda_0 s) + \frac{1}{2}(a\nu)^2\cos \omega_s s = R_n(s) + R_s(s). \tag{16}$$

Formula (16) is simpler than its counterpart in [1]. It is obtained by noting that the terms of order ν^2 are substantial only for the periodic component $R_s(s)$ and can be ignored in the aperiodic component $R_n(s)$. The associated power spectrum is the superposition of the continuous noise spectrum $S_n(\Omega)$ and the discrete spectrum $S_s(\Omega)$, namely

$$S(\Omega) = S_n(\Omega) + S_s(\Omega),$$

$$S_n(\Omega) = \frac{4a^2\lambda_0}{4\lambda_0^2 + \Omega^2}, \tag{17}$$

$$S_s(\Omega) = +2\pi(a\nu)^2[\delta(\Omega - \omega_s) + \delta(\Omega + \omega_s)].$$

It is easy to see that there is no correlation between the form of the flat spectrum $S_n(\Omega)$ and the signal frequency ω_s. Signal-to-noise ratio at the frequency ω_s is defined

$$\eta(\sigma) = \frac{2\pi(a\nu)^2}{S_n(\omega_s)} = \pi\nu^2\frac{4\lambda_0^2 + \omega_s^2}{2\lambda_0}. \tag{18}$$

Signal-to-noise ratio for the slow signal with the frequency $\omega_s \ll 2\lambda_0$,

$$\eta(\sigma) = 2\pi\nu^2\lambda_0, \tag{19}$$

is independent of the frequency ω_s. Using the expressions for λ_0 and ν, we obtain

$$\eta(\sigma) = 8\pi I_s\omega_0\left(\frac{\beta^2\gamma a}{\sigma^3}\right)^2\exp\left[\frac{-2\beta\Delta U}{\varepsilon\sigma^2}\right]. \tag{20}$$

The maximum of $\eta(\sigma)$ is defined by the equation $d\eta(\sigma)/d\sigma=0$. The direct calculation gives the optimal noise intensity

$$\sigma^{*} = (2\varepsilon^{-1}\beta\Delta U/3)^{1/2}. \tag{21}$$

We have demonstrated that there is no direct correlation between the signal frequency and the switching rate, and optimum tuning is independent of the signal frequency.

Compare functions (20) and (21) with their counterparts based on the Kramers rate (5). Using [1], we obtain after the proper scaling

$$\eta^{\alpha}(\sigma) = \frac{\sqrt{2}\omega_0\omega_1 a^2\beta^3}{\varepsilon\sigma^4}\exp\left[\frac{-2\beta\Delta U}{\varepsilon\sigma^2}\right]. \tag{22}$$

The optimal noise intensity is defined by the equation $d\eta^{\alpha}(\sigma)/d\sigma=0$. The result is

$$\sigma^{\alpha} = (\varepsilon^{-1}\beta\Delta U)^{1/2} \approx 1{,}2\sigma^{*}. \tag{23}$$

On the logarithmic scale we have $\ln \eta(\sigma)\to\ln \eta^{\alpha}(\sigma)$ as $\varepsilon\to 0$. Thus the curves $\eta(\sigma)$ and $\eta^{\alpha}(\sigma)$ and the optimal values σ^{*} and σ^{α} are close enough from the modeling and analysis viewpoints. This implies that a two-state model with the Kramers escape rate is convenient for describing the SR effect in a fairly general class of bistable systems.

The theory predicts $\eta(\sigma)\to 0$ for very large and very low input noise, and yet it is clear from Fig. 9 in [1] that $\eta(\sigma)\to\infty$ as $\sigma\to 0$. This effect can be easily understood. The signal alone is insufficient to induce transitions across the potential barrier but the weak noise helps to bring about such transitions. If noise is too small, the escape rate is close to zero, that is motion is confined near the bottom of a single well. In this case the system dynamics is quasilinear, and any increase in the input noise would result in a proportional decrease in the output SNR. As the input noise increases, the hopping dynamic becomes dominant. Nonlinear interaction between signal and noise can, under certain conditions, result in an improvement of the output SNR as the input noise increases. A further increase in the input noise suppresses the signal effect on the system, and the output SNR decreases. If σ is large enough, irregular hopping is transformed into random oscillation overlapping both stable positions. Hence, SNR may pass through a maximum at an optimal level of noise. Stochastic resonance can be interpreted as a nonlinear effect arising due to the passage from oscillatory to noise-induced hopping motion. In particular, this implies that the SNR curve cannot be extrapolated to the margins $\sigma \to 0$ and $\sigma \to \infty$, as it has been done in a number of issues (see [2] for references).

IV. NOISE-INDUCED SYNCHRONIZATION IN A WEAKLY PERTURBED SYSTEM

A. Synchronized transitions

The system is said to be weakly perturbed if the input SNR $\nu > 1$. In this case the mean escape rate can be found from the asymptotic representation of the modified Bessel functions [10]

$$I_n(\nu) \sim (2\pi\nu)^{-1/2} e^{\nu}[1 + O(\nu^{-1})] \tag{24}$$

for all $n \geq 1$, $\nu > 1$. From formulas (10) and (24) we find

$$\lambda^+(\tau) = l_0(\nu)\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \sin n\tau\right],$$

$$\lambda^-(\tau) = l_0(\nu)\left[1 + 2\sum_{n=1}^{\infty} \sin n\tau\right], \tag{25}$$

where $l_0(\nu) = \lambda_0(2\pi\nu)^{-1/2}e^{\nu}$. Summing series (25) as distributions, we obtain

$$\lambda^+(\tau) = l_0(\nu)\sum_{n=1}^{\infty} \delta(\tau - (2k-1)\pi),$$

$$\lambda^-(\tau) = l_0(\nu)\sum_{n=1}^{\infty} \delta(\tau - 2k\pi), \tag{26}$$

where $\delta(\tau)$ is Dirac's δ function. From (24), (26) we deduce that, with probability close to 1, a particle rests at the bottom of the deepest well until the wells switch, and the bottom of the well takes up the highest position. The moments of escape from the right (+) and left (−) wells are, respectively

$$t_k^+ = (2k-1)\pi/\omega_s, \quad t_k^- = 2k\pi/\omega_s, \quad k = 1,2,\ldots. \tag{27}$$

It follows from (24), (26), (27) that hopping can be interpreted as a series of $2\pi/\omega_s$—periodic switching between the states $\pm a$, that is

$$x(t) = -a, \quad t_k^+ < t < t_k^-; \quad x(t) = a, \quad t_k^- < t < t_{k+1}^+. \tag{28}$$

Formulas (25)–(28) describe the phenomenon of noise-induced synchronization. Interwell jumps are induced by noise but motion becomes "captured" and sustained by a relatively strong signal. The hopping frequency coincides with the signal frequency.

Relations (11) and (25) imply that the steady-state probability $\mathbf{P}^{\pm}(t)$ can be approximated as follows:

$$\mathbf{P}^+(t) = \{1, \text{ if } \sin\omega_s t > 0; 0, \text{ if } \sin\omega_s t < 0\},$$

$$\mathbf{P}^-(t) = \{0, \text{ if } \sin\omega_s t > 0; 1, \text{ if } \sin\omega_s t < 0\}. \tag{29}$$

Hence a particle, escaped eventually from a well, is captured into the deepest well with the probability close to 1 and then continues moving by law (28).

Actually, we present an idealized depiction of the synchronized transitions. Noise induces an eventual excursion from a deeper well to a higher well, or escapes from a higher well at the moments different from $t_k^{\pm}$. Due to these random walks, the spectrum of hopping is continuous, with sharp peaks at the signal harmonics. The intensity of these random walks and the portion of the wide-band spectrum depend on the relationship between the input signal and noise.

B. Upper and lower bounds of noise-induced synchronization

If $\nu \ll 1$, then interwell jumps may occur at random moments during each semicycle of modulation, and the signal effect is almost negligible. As the relative level of the signal against noise is increasing, the wide-band portion of the spectrum decreases, and locking of the hopping process to a weak signal becomes visible. This yields the upper bound of the noise intensity allowing synchronization $\nu = 2\beta a\gamma/\sigma^2 > 1$, or

$$\sigma^2/2\beta < a\gamma. \tag{30}$$

Let $T_s = 2\pi/\omega_s$ be the signal period. If escape time from a well is $T^\varepsilon < T_s/2 = \pi/\omega_s$, the process has enough time to escape from the higher well and to enter the lower well until the bottom of the well where it starts is in the higher position. In the opposite case $T^\varepsilon > T_s/2 = \pi/\omega_s$, the process does not have enough time to escape until the bottom of the well is in the higher position, so it remains confined within a single well with high probability. Hence only below the critical level $T^\varepsilon = \pi/\omega_s$ the periodic hopping (28) can occur. This entails the lower bound of synchronization $T^\varepsilon < \pi/\omega_s$.

In principal, T^ε is a random parameter. However, with probability close to 1 we have $T^\varepsilon \to \Theta^\varepsilon = \mathbf{E}T^\varepsilon$ as $\varepsilon \to 0$ [4]. In turn, from formula (4) we have $\Theta^\varepsilon \to \theta^0$ as $\varepsilon \to 0$. This yields the deterministic estimate $\theta^0 < \pi/\omega_s$, or, by (4),

$$\sigma^2/2\beta > \Delta U/[\varepsilon \ln(1/\omega_s)]. \tag{31}$$

Assumption (6) implies $\ln(1/\omega_s) \to k/\varepsilon$ as $\varepsilon \to 0$, that is

$$\sigma^2/2\beta > \Delta U/k. \tag{32}$$

Inequalities (30) and (32) define the interval of noise-induced synchronization

$$\Delta U/k < \sigma^2/2\beta < \gamma a. \tag{33}$$

V. CONCLUSIONS

In this paper we have shown that the phenomena of stochastic resonance and noise-induced synchronization are different but not contradictory. These effects can be interpreted as the limit cases of interwell transitions modulated by a weak signal. Stochastic resonance is not directly correlated with the matching of the signal frequency and the switching rate. The boundary between the domains of stochastic resonance and synchronization depends on the input signal-to-noise ratio. If the signal is weak compared to noise, the hopping dynamics is random, with a weak periodic component, and, under certain conditions, stochastic resonance can appear. While the relative signal intensity increases, the wide-band portion of the spectrum decreases, and, in the limit, a nearly periodic hopping occurs. Nearly periodic hopping induced by noise but locked to a periodic signal is interpreted as noise-induced synchronization. We have deduced the upper and lower bounds of the noise intensity, allowing noise-induced synchronization.

Large deviations theory reduces the hopping dynamics in a bistable system to a simple Poissonian switching model. This model depends on the system's structure only through the mean escape rate of the original system. We have shown that the Kramers rate of the first-order system is less sensitive to the noise intensity than the escape rate of the second-order oscillator but both parameters are qualitatively similar. This implies that a simple two-state Poissonian model, based on the Kramers escape rate, gives a fairly good approximation in the analysis of a general class of bistable oscillators. This deduction is confirmed by the similarity of the SR curves for the first-order and second-order models.

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[1] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[2] For review see K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995); A. Bulsara and L. Gammaitoni, Phys. Today **49** (3), 39 (1996); L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchezoni, Rev. Mod. Phys. **70**, 223 (1998); *Stochastic and Chaotic Dynamics in the Lakes*, edited by S. Broomhead *et al.* (AIP, Melville, NY, 2000); an extensive bibliography can be found at http://www.pg.infin.it/sr

[3] M. Freidlin, Physica D **137**, 313 (2000); J. Stat. Phys. **103**, 283 (2001); P. Imkeller and I. Pavljukevich, Stochastics Dyn. **2**, 463 (2002).

[4] M. Freidlin and A. Wentzell, *Random Perturbations of Dynamical Systems*, 2nd ed. (Springer-Verlag, Berlin, 1998).

[5] B. Shulgin, A. Neiman, and V. Anishchenko, Phys. Rev. Lett. **75**, 4157 (1995); M. Tretyakov, *Numerical Studies of Stochastic Resonance* (Weierstrass-Institute fur Angewandte Analysis and Stochastic, Berlin, 1997).

[6] Yu. L. Klimontovich, Phys. Usp. **42**, 37 (1999).

[7] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1986).

[8] A. Kovaleva, in *Proceedings of the IUTAM Symposium on Chaotic Dynamics and Control of Systems and Processes in Mechanics*, edited by G. Rega and F. Vestroni (Springer-Verlag, Berlin, 2005).

[9] C. W. Gardiner, *Handbook of Stochastic Methods*, 3rd ed. (Springer-Verlag, Berlin, 2004).

[10] I. S. Gradstein and I. M. Ryshik, *Tables of Series, Products, and Integrals*, 6th ed. (Academic Press, New York, 2000).